

# IDEAS OF E. CARTAN AND S. LIE IN MODERN GEOMETRY: G-STRUCTURES AND DIFFERENTIAL EQUATIONS. LECTURE 4

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## Problem:

How to find invariants of singularities of a  $G$ -structure?

## SIMPLE EXAMPLE

Let  $V = (v^1, v^2)$  be a vector field on  $\mathbb{R}^2$ . If  $V(x_0) \neq 0$ , then  $V$  restricted to a neighborhood  $U(x_0)$  is equivalent to  $\partial_1$ , hence all nonvanishing vector fields are locally equivalent and there are no invariants.

If  $V(x_0) = 0$ , and  $W(x_0) = 0$ , and the matrix  $\|\partial_i V^j(x_0)\|$  is not similar to the matrix  $\|\partial_i W^j(x_0)\|$ , then  $V$  and  $W$  are not locally equivalent and there arise invariants.

Generally, *non-regular points have their own invariants.*

## 1. JETS OF SMOOTH MAPS

Let  $M$  and  $N$  be smooth manifolds,  $x \in M$  is a point. We will denote by  $f : (M, x) \rightarrow N$  a smooth map which is defined in an open neighborhood  $U$  of  $x$ . By  $D(f)$  we will denote the domain  $U$ .

Denote by  $C^\infty((M, x), N)$  the set of all smooth maps  $f : (M, x) \rightarrow N$ . On the set  $C^\infty((M, x), N)$  we introduce the equivalence relation: we say that  $f, g$  in  $C^\infty((M, x), N)$  are equivalent at  $x \in M$  ( $f \sim_x g$ ) if

$$(1) \quad \exists W \in \mathcal{U}_x, \quad W \subset D(f) \cap D(g) \text{ such that } f|_W = g|_W.$$

**Definition 1.** A *germ of a map*  $f \in C^\infty((M, x), N)$  at the point  $x$  is the equivalence class of  $f \in C^\infty((M, x), N)$  with respect to  $\sim_x$  denoted by  $\langle f \rangle_x$ .

We set

$$(2) \quad \mathcal{G}_x(M, N) = C^\infty((M, x), N) / \sim_x = \{ \langle f \rangle_x \mid f \in C^\infty((M, x), N) \}.$$

If  $M_1, M_2$ , and  $M_3$  are smooth manifolds, and  $f_1 \in C^\infty((M_1, x_1), M_2)$ ,  $f_2 \in C^\infty((M_2, x_2), M_3)$  with  $f_1(x_1) = x_2$ , then we can define the composition of germs as follows:

$$(3) \quad \langle f_2 \rangle_{x_2} \circ \langle f_1 \rangle_{x_1} = \langle f_2 \circ f_1 \rangle_{x_1}$$

Another equivalence relation  $\sim_k$  on the set  $C^\infty((M, x), N)$  is introduced as follows: for  $f, g \in C^\infty(M, N)$  such that  $f(x) = g(x)$  we take the coordinate systems  $(U, x^i)$  in a neighborhood  $U$  of  $x$  and  $(V, y^\alpha)$  in a neighborhood  $V$  of  $y$ . Then we say that  $f$  and  $g$  are equivalent ( $f \sim_k g$ ) if

the Taylor series of the coordinate representations of  $f$  and  $g$  coincide up to the order  $k$ . One can prove that this equivalence relation does not depend on the choice of coordinate systems.

**Definition 2.** The equivalence class  $j_x^k f$  of  $f$  is called *k-jet of the map  $f$  at the point  $x$* .

The set of all  $k$ -jets of maps  $f \in C^\infty((M, x), N)$  will be denoted by  $J_x^k(M, N)$ .

It is clear that if  $f, g \in C^\infty((M, x), N)$  determine the same germ at  $x$ , that is if  $f \sim_x g$ , then  $j_x^k f = j_x^k g$  for any  $k$ . Also, the composition of maps, or the composition of germs, define the composition of  $k$ -jets:

$$(4) \quad j_{f_1(x)}^k(f_2) \circ j_x^k(f_1) = j_x^k(f_2 \circ f_1).$$

For the set of all  $k$ -jets  $J^k(M, N)$  one can define two natural projections:

$$(5) \quad \pi_0 : J^k(M, N) \rightarrow M, \quad j_x^k f \mapsto x,$$

$$(6) \quad \pi_1 : J^k(M, N) \rightarrow N, \quad j_x^k f \mapsto f(x).$$

The set  $J^k(M, N)$ ,  $\dim M = m$ ,  $\dim N = n$  can be endowed by a manifold structure so that these projections are bundle projections. We set

$$(7) \quad J_{x,y}^k(M, N) = \{j_x^k f \in J^k(M, N) \mid \pi_0(j_x^k f) = x, \pi_1(j_x^k f) = y\}.$$

The typical fiber of the bundle  $(J^k(M, N), \pi_0, M)$  is the manifold  $J_{0,0}^k(\mathbb{R}^m, \mathbb{R}^n) \times N$ , and of the bundle  $(J^k(M, N), \pi_1, N)$  is  $J_{0,0}^k(\mathbb{R}^m, \mathbb{R}^n) \times M$ .

The manifold  $J^k(M, M)$  endowed with the projections  $\pi_0$  and  $\pi_1$ , and the composition of  $k$ -jets is a groupoid.

If  $\pi : E \rightarrow M$  is a fiber bundle, then the set of  $k$ -jets of sections of the bundle  $E$  is denoted by  $J^k(E)$ .

## 2. BUNDLE OF GERMS OF DIFFEOMORPHISMS. COFRAME BUNDLE OF $k$ TH ORDER

**2.1. Bundle of germs of diffeomorphisms.** Let  $\mathcal{D}(m)$  be the group of all germs of local diffeomorphisms of  $\mathbb{R}^m$  at  $0 \in \mathbb{R}^m$ :

$$\mathcal{D}(m) = \{\langle \varphi \rangle_0 \mid \varphi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0) \text{ is a local diffeomorphism}\}$$

endowed with the operation of composition of germs:  $\langle \varphi_1 \rangle_0 \circ \langle \varphi_2 \rangle_0 = \langle \varphi_1 \circ \varphi_2 \rangle_0$ .

Now let us consider

$$\mathcal{B}(M) = \{\langle f \rangle_x \mid f : (M, x) \rightarrow (\mathbb{R}^n, 0) \text{ is a local diffeomorphism}\}$$

We have natural projection

$$\pi : \mathcal{B}(M) \rightarrow M, \quad \pi(\langle f \rangle_x) = x,$$

and the natural right action of  $\mathcal{D}(m)$  on  $\mathcal{B}(M)$ :

$$\langle f \rangle_x \cdot \langle \varphi \rangle_0 = \langle \varphi^{-1} \circ f \rangle_x.$$

so  $(\mathcal{B}(M), \pi, M)$  can be considered as a “principal fiber bundle”.

Let  $(U, u : U \rightarrow V \subset \mathbb{R}^m)$  be a coordinate map. This map determines the “trivialization” of the bundle  $\mathcal{B}(M)$  over  $U$ . Let

$$t_a : \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad t_a(v) = v + a$$

be the parallel translation of  $\mathbb{R}^m$ . Then

$$\mathcal{U} : \pi^{-1}(U) \rightarrow U \times \mathcal{D}(m), \quad \langle f \rangle_x \rightarrow (x, \langle f \circ u^{-1} \circ t_{u(x)} \rangle_0).$$

gives us the required trivialization. The inverse map is

$$\mathcal{U}^{-1} : U \times \mathcal{D}(m) \rightarrow \pi^{-1}(U) \quad (x, \langle \varphi \rangle_0) \rightarrow \langle t_{-u(x)} \circ u \circ f^{-1} \rangle.$$

Now assume that we have two coordinate systems  $(U, u)$  and  $(U, \bar{u})$  on  $M$ . Then,

$$\bar{\mathcal{U}} \circ \mathcal{U}^{-1} : U \times \mathcal{D}(m) \rightarrow U \times \mathcal{D}(m), \quad (x, \varphi) \rightarrow (x, t_{-\bar{u}(x)} \circ \bar{u} \circ u^{-1} \circ t_{u(x)}).$$

so the gluing functions of the atlas of the “principal bundle”  $(\mathcal{B}(M), \pi, M)$  constructed by an atlas  $(U_\alpha, u_\alpha)$  of the manifold  $M$  are

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \mathcal{D}(m), \quad g_{\beta\alpha}(x) = t_{-u_\beta(x)} \circ u_\beta \circ u_\alpha^{-1} \circ t_{u_\alpha(x)}.$$

**Remark 1.** In what follows we will use unordered multiindices. We denote by  $\mathcal{I}(m)$  the set of all unordered multiindices  $I = \{i_1 i_2 \dots i_k\}$ , where  $1 \leq i_l \leq m$ , for all  $l = \overline{1, k}$ . The number  $k$  is called the length of the multiindex and is denoted by  $|I|$ . Also, we set  $I_k(m) = \{I \in \mathcal{I}(m) \mid |I| = k\}$ .

**2.2. Differential group of  $k$ th order.** The  $k$ th order *differential group* is the set of  $k$ -jets:

$$D^k(m) = \{j_0^k(\varphi) \mid \varphi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0) \text{ is a local diffeomorphism} \}.$$

On the set  $D^k(m)$  consider the operation

$$(8) \quad D^k(m) \times D^k(m) \rightarrow D^k(m), \quad j_0^k(\varphi) \cdot j_0^k(\psi) = j_0^k(\varphi \circ \psi),$$

then  $(D^k(m), \cdot)$  is a group.

Denote  $\varphi_I^k = \partial_I|_0 \varphi^k$ . Then

$$(9) \quad \mathcal{C}^k : D^k(m) \rightarrow \mathbb{R}^N, \quad j_0^k(\varphi) \rightarrow \{\varphi_I^k\}$$

in a one-to-one map of  $D^k(m)$  onto the open set in  $\mathbb{R}^N$  determined by the inequality  $\det \|\varphi_i^k\| \neq 0$ . In this way we get globally defined coordinates on  $D^k(m)$  which will be called *natural coordinates*. With respect to the natural coordinates the product (8) is written in terms of polynomials, therefore is a smooth map. Thus  $D^k(m)$  is a Lie group.

**2.3. Bundle of  $k$ th order holonomic coframes.** For an  $m$ -dimensional manifold  $M$  consider the set

$$B^k(M) = \{j_x^k f \mid f : (M, x) \rightarrow (\mathbb{R}^m, 0) \text{ is a local diffeomorphism} \}$$

whose elements are called  *$k$ -coframes* or *coframes of order  $k$*  of the manifold  $M$ . We have the projection

$$\pi^k : B^k(M) \rightarrow M, \quad \pi(j_x^k(f)) = x.$$

On the set  $B^k(M)$  we have the right  $D^k(m)$ -action:

$$j_x^k(f) \cdot j_0^k(\varphi) = j_0^k(\varphi^{-1} \circ f).$$

and one can easily prove that this action is free and its orbits are the fibers of the projection  $\pi$ .

2.3.1. *Trivializing charts of  $B^k(M)$ . Gluing maps.* In what follows we set  $t_a : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $t_a(x) = x + a$ , the parallel translation of  $\mathbb{R}^m$  with respect to  $a \in \mathbb{R}^m$ .

Let  $(U, u : U \rightarrow V \subset \mathbb{R}^m)$  be a coordinate chart on  $M$ . We have the one-to-one map

$$(10) \quad \mathcal{T}^k : (\pi^k)^{-1}(U) \rightarrow U \times D^k(m),$$

$$j_x^k(f) \rightarrow (x, j_0^k(t_{-u(x)} \circ u \circ f^{-1})).$$

The map  $\mathcal{T}^k$  is  $D^k(m)$ -equivariant because

$$\begin{aligned} \mathcal{T}^k(j_x^k f \cdot j_0^k \varphi) &= \mathcal{T}^k(j_x^k(\varphi^{-1} \circ f)) = (x, j_0^k(t_{-u(x)} \circ u \circ f^{-1} \circ \varphi)) = \\ &= (x, j_0^k(t_{-u(x)} \circ u \circ f^{-1}) \cdot j_0^k \varphi) = (x, j_0^k(t_{-u(x)} \circ u \circ f^{-1})) \cdot j_0^k \varphi. \end{aligned}$$

Since  $D^k(m)$  is a Lie group, the map  $\mathcal{T}^k$  defines a trivializing chart for the map  $\pi^k : B^k(M) \rightarrow M$ .

Therefore, for each atlas  $(U_\alpha, u_\alpha)$ , we construct the atlas of trivializing charts  $(U_\alpha, \mathcal{T}_\alpha^k)$ . Find the gluing maps for this atlas.

Assume that  $(U_\alpha, u_\alpha)$ ,  $(U_\beta, u_\beta)$  are two coordinate systems on  $M$ , and  $U_\alpha \cap U_\beta \neq \emptyset$ . Then

$$j_0^k(t_{-u_\beta(x)} \circ u_\beta \circ f^{-1}) = j_0^k(t_{-u_\beta(x)} \circ u_\beta \circ u_\alpha^{-1} \circ t_{u_\alpha(x)}) \cdot j_0^k(t_{-u_\alpha(x)} \circ u_\alpha \circ f^{-1})$$

Therefore, the gluing maps are

$$(11) \quad g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow D^k(m), \quad g_{\beta\alpha}(x) = j_0^k(t_{-u_\beta(x)} \circ u_\beta \circ u_\alpha^{-1} \circ t_{u_\alpha(x)})$$

Since the gluing functions are smooth, we conclude that  $\pi^k : B^k(M) \rightarrow M$  is a  $D^k(m)$ -principal bundle over  $M$  which is called *the bundle of  $k$ -coframes of  $M$*  or *the bundle of coframes of order  $k$  of  $M$* .

For a coordinate chart  $(U, u)$  there is defined a section

$$(12) \quad s : U \rightarrow B^k(M), \quad s(x) = j_x^k u,$$

which is called the *natural  $k$ -coframe field* associated with a coordinate chart  $(U_\alpha, u_\alpha)$ .

2.3.2. *Natural coordinates on  $B^k(M)$ .* If  $(U, u)$  is a coordinate chart on  $M$ , and  $\mathcal{T}^k$  is the corresponding trivialization of  $B^k(M)$ . Then

$$(13) \quad (u \times \mathcal{C}^k) \circ \mathcal{T}^k : (\pi^k)^{-1}(U) \rightarrow \mathbb{R}^m \times \mathbb{R}^N$$

gives *natural local coordinates on  $B^k(M)$* .

The section  $s : U \rightarrow B^k(M)$  (12) is written with respect to this coordinate system as follows:

$$(14) \quad s(u^k) = (u^k, \delta_i^k, 0).$$

**Remark 2.** We have the natural projections  $\pi_l^k : B^k(M) \rightarrow B^l(M)$ ,  $k \geq l$ , which are, in turn, principal fiber bundles with the group  $H_l^k$  which is the kernel of the natural homomorphism  $D^k(m) \rightarrow D^l(m)$ .

2.4. **Case  $k = 1$ .** The Lie group  $D^1(m) \cong GL(m)$  and  $B^1(M) = B(M)$  is the coframe bundle of  $M$ .

2.5. **Case  $k = 2$ .**

2.5.1. *The group  $D^2(m)$ .* Elements of the group  $D^2(m)$  are the 2-jets of germs  $\varphi \in \mathcal{D}(m)$ . The coordinate system (9) in this case is

$$(15) \quad j_0^2 \varphi \longrightarrow (\varphi_i^k, \varphi_{ij}^k), \text{ where } \varphi_i^k = \frac{\partial \varphi^k}{\partial u^i}(0), \quad \varphi_{ij}^k = \frac{\partial^2 \varphi^k}{\partial u^i \partial u^j}(0).$$

Here  $u^i$  are coordinates on  $\mathbb{R}^m$ , and it is clear that  $\varphi_{ij}^k = \varphi_{ji}^k$ . From this follows that  $\dim D^2(m) = m^2 + m^2(m+1)/2$ .

Now, if  $j_0^2 \varphi \rightarrow (\varphi_i^k, \varphi_{ij}^k)$ ,  $j_0^2 \psi \rightarrow (\psi_i^k, \psi_{ij}^k)$ , and

$$j_0^2 \psi \cdot j_0^2 \varphi = j_0^2(\psi \circ \varphi) \rightarrow (\eta_i^k, \eta_{ij}^k),$$

by the chain rule we get that

$$(16) \quad \eta_i^k = \psi_s^k \varphi_i^s, \quad \eta_{ij}^k = \psi_{pq}^k \varphi_i^p \varphi_j^q + \psi_s^k \varphi_{ij}^s$$

These formulas express the product in the group  $D^2(m)$  in terms of the natural coordinates  $(\varphi_i^k, \varphi_{ij}^k)$ .

2.5.2. *The bundle  $B^2(M)$ .* The elements of  $B^2(M)$  are the 2-jets of local diffeomorphisms  $f : (M, x) \rightarrow (\mathbb{R}^m, 0)$ . The natural coordinates (14) in this case can be found as follows. Let  $(U, u)$  be a coordinate chart on  $M$ . Then, for any  $j_x^2 f$  with  $x \in U$ , the diffeomorphism

$$f \circ u^{-1} : (\mathbb{R}^m, u(x)) \rightarrow (\mathbb{R}^m, 0)$$

can be written as  $w^k = f^k(u^i)$ , where  $w^k$  are standard coordinates on  $\mathbb{R}^m$ , and  $f^k(u^i(x)) = 0$ . Then take the inverse diffeomorphism  $u^k = \tilde{f}^k(w^i)$ , and the local diffeomorphism  $t_{-u(x)} \circ u \circ f^{-1}$  has the form  $\tilde{f}^k(w^i) - u^k(x)$ . Therefore, the natural coordinates of  $j_x^k f$  induced by a coordinate chart  $(U, u)$  on  $M$  are  $(u^k, u_i^k, u_{ij}^k)$ , where

$$(17) \quad u_i^k = \frac{\partial \tilde{f}^k}{\partial w^i}(0), \quad u_{ij}^k = \frac{\partial^2 \tilde{f}^k}{\partial w^i \partial w^j}(0).$$

The derivatives of  $\tilde{f}$  at 0 can be expressed in terms of the derivatives of  $f$  at  $u^k(x)$ . If we denote

$$f_i^k = \frac{\partial f^k}{\partial u^i}(u(x)), \quad f_{ij}^k = \frac{\partial^2 f^k}{\partial u^i \partial u^j}(u(x)).$$

then

$$u_i^k = \tilde{f}_i^k, \quad u_{ij}^k = -\tilde{f}_s^k f_{lm}^s \tilde{f}_i^l \tilde{f}_j^m.$$

With respect to the natural coordinate system the  $D^2(m)$ -action is written as follows:

$$(18) \quad (u^k, u_i^k, u_{ij}^k) \cdot (\varphi_i^k, \varphi_{ij}^k) = (u^k, u_s^k \varphi_i^s, u_{pq}^k \varphi_i^p \varphi_j^q + u_s^k \varphi_{ij}^s).$$

Let us express the gluing maps in terms of the natural coordinates. If  $(U, u)$  and  $(U', u')$  are coordinate charts on  $M$  such that  $U \cap U' \neq \emptyset$ , then from (11) it follows that the corresponding gluing map is

$$g : U \cap U' \rightarrow D^2(m), \quad g(x) = j_0^2(t_{-u'(x)} \circ u' \circ u^{-1} \circ t_{u(x)})$$

If the coordinate change  $u' \circ u^{-1}$  is written as  $v^k = v^k(u^i)$ , then we have to take derivatives at 0 of the map  $v^k(u^i + u^i(x)) - v^k(u^i(x))$ , which are equal to the derivatives of the functions  $v^k$  at  $u^k(x)$ . Therefore,

$$(19) \quad g : U \cap U' \rightarrow D^2(m), \quad g(x) = \left( \frac{\partial v^k}{\partial u^i}(u(x)), \frac{\partial v^k}{\partial u^i \partial u^j}(u(x)) \right).$$

Therefore, the bundle  $B^2(M) \rightarrow M$  is the  $D^2(m)$ -principal bundle with gluing maps (19).

### 3. FIRST PROLONGATION OF A $G$ -STRUCTURE

**3.1. First prolongation of an integrable  $G$ -structure.** Let  $P(M) \rightarrow M$  be an integrable  $G$ -structure, that is a subbundle of  $B(M)$  such that there exists an atlas  $\mathcal{A} = (U_\alpha, u_\alpha)$  such that the natural coframes of the atlas are sections of  $P(M)$ , or equivalently, the coordinate change  $u^{k'} = u^{k'}(u^i)$  has the property that  $\|\frac{\partial u^{k'}}{\partial u^k}\| \in G$ .

In this case, we can specify the set  $\mathcal{B}_G$  of local diffeomorphisms  $f : (M, x) \rightarrow (\mathbb{R}^m, 0)$  such that for each coordinate map  $u$  of the atlas  $\mathcal{A}$ , the local diffeomorphism  $f \circ u^{-1}$  has the Jacobi matrix at in  $G$  at all points of its domain. It is clear that

$$(20) \quad P(M) = \left\{ j_x^1 f \mid \left\| \frac{\partial (f \circ u^{-1})^k}{\partial u^i} \Big|_{u(x)} \right\| \in G \right\}$$

and consider

$$(21) \quad P^1(M) = \{ j_x^2 f \mid f \in \mathcal{B}_G \}$$

**Remark 3.** Note that if  $j_x^2 f$  is an element of  $P^1(M)$ , then the Jacobi matrix  $\left\| \frac{\partial (f \circ u^{-1})^k}{\partial u^i} \Big|_{u(x)} \right\|$  is an element of  $G$ . However, the converse is not true because, by definition, the Jacobi matrix belongs to  $G$  for each point which implies conditions on the second derivatives of  $f \circ u^{-1}$  (see below).

In the same manner introduce the set  $\mathcal{D}_G$  of local diffeomorphisms  $\varphi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$  whose Jacobi matrices are elements of  $G$  at all points of their domains. Consider the Lie subgroup of  $D^2(m)$ :

$$(22) \quad G^1 = \{ j_0^2 \varphi \in D^2(m) \mid \varphi \in \mathcal{D}_G \}$$

The Lie subgroup  $G^1 \subset D^2(m)$  is called the *first holonomic prolongation of the group  $G$* .

The subset  $P^1(M)$  is a submanifold of  $B^2(M)$ , and is the total space of a principal subbundle of  $B^2(M) \rightarrow M$  with the subgroup  $G^1 \subset D^2(m)$ . This subbundle is called the *first holonomic prolongation of the integrable  $G$ -structure  $P$* .

It is also clear that, for the atlas  $\mathcal{A}$  the gluing map (19) takes values in the subgroup  $G^1$ . Therefore, an integrable  $G$ -structure defines reduction of the principal bundle  $B^2(M)$  to the structure group  $G^1 \subset D^2(m)$ .

**3.2. Algebraic structure of the Lie group  $G^1$ .** We have the surjective Lie group morphism

$$(23) \quad p^1 : G^1 \rightarrow G, \quad j_0^2 \varphi \mapsto j_0^1 \varphi.$$

Let us find the kernel of  $p^1$ . Let  $\varphi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$  be an element of  $\mathcal{D}_G$ . Then, the map  $g(u^k) = \|\frac{\partial \varphi^k}{\partial u^i}\|$  takes values in  $G$ , and  $g(0) = I \in G$ . Therefore,

$$\left. \frac{\partial^2 \varphi}{\partial u^i \partial u^j} \right|_0 = \left. \frac{\partial g_i^k}{\partial u^j} \right|_0$$

is a linear map  $t : \mathbb{R}^m \cong T_0 \mathbb{R}^m \rightarrow \mathfrak{g}(G) \subset \mathfrak{gl}(m)$  but with property that  $t(u)v = t(v)u$ . In other words, elements of  $\ker p^1$  are tensors of type  $(2, 1)$  on  $\mathbb{R}^m$  such that  $t_{ij}^k \in \mathfrak{g}$  for each  $i$ , and  $t_{ij}^k = t_{ji}^k$ . The vector space of such tensors is called the *first prolongation of the Lie algebra*  $\mathfrak{g}$  and is denoted by  $\mathfrak{g}^1$ . From this follows that  $\ker p^1$  is a commutative Lie group.

Thus we have the exact sequence of Lie groups

$$0 \rightarrow \mathfrak{g}^1 \rightarrow G^1 \rightarrow G \rightarrow e,$$

This sequence admits a splitting: for any  $\|g_i^k\| \in G$  we take the diffeomorphism  $\varphi^k(u^i) = g_i^k u^i$ , which is evidently lies in  $\mathcal{D}_G$ , and set  $s(j_0^1 \varphi) = j_0^2 \varphi$ .

According to the group theory, the exact splitting sequence (3.2) determines the right action of  $G$  on  $\mathfrak{g}^1$ :  $R_g t = s(g^{-1}) \cdot t \cdot s(g)$ , and  $G^1$  is the extension of  $G$  by the commutative group  $\mathfrak{g}^1$  with respect to the action  $R$ . This means that we have the group isomorphism

$$(24) \quad G^1 \rightarrow G \times \mathfrak{g}^1, \quad g^1 \mapsto (p^1(g^1), s(p^1((g^1)^{-1}))g^1,$$

therefore

$$(25) \quad G^1 \cong G \times \mathfrak{g}^1, \text{ and } (g_1, t_1) \cdot (g_2, t_2) = (g_1 g_2, R(g_2)t_1 + t_2).$$

Let us express this representation of  $G^1$  in terms of the canonical coordinates. We have  $p^1(\varphi_i^k, \varphi_{ij}^k) = \varphi_i^k$ , and  $s(\varphi_i^k) = (\varphi_i^k, 0)$ . Hence follows that the isomorphism (24) is

$$(\varphi_i^k, \varphi_{ij}^k) \rightarrow (g_i^k, t_{ij}^k) \text{ with } g_i^k = \varphi_i^k, \quad t_{ij}^k = \tilde{\varphi}_s^k \varphi_{ij}^s.$$

Thus we get *algebraic coordinates*  $(g_i^k, t_{ij}^k)$  on  $G^1$  which are adopted to the algebraic structure of  $G^1$ .

The right action  $R$  with respect to the canonical coordinates is written as follows (we use (16)):

$$(\tilde{\varphi}_s^k, 0) \cdot (\delta_i^k, \varphi_{ij}^k) \cdot (\varphi_i^k, 0) = (\delta_i^k, \tilde{\varphi}_s^k \psi_{lm}^s \varphi_i^l \varphi_j^m).$$

At the same time, the algebraic coordinates of the elements  $(\varphi_i^k, 0)$  and  $(\delta_i^k, 0)$  are the same, this means that they are  $(g_i^k = \varphi_i^k, 0)$  and  $(\delta_i^k, t_{ij}^k = \varphi_{ij}^k)$ . Therefore, with respect to the algebraic coordinates the product of the group  $G^1$  looks like (see (25)):

$$(26) \quad (g_i^k, t_{ij}^k) \cdot (h_i^k, q_{ij}^k) = (g_s^k h_i^s, \tilde{h}_s^k t_{pq}^s h_i^p h_j^q + q_{ij}^k).$$

**3.3. Algebraic coordinates on  $B^2(M)$ . Description of  $P^1(M)$  with respect to algebraic coordinates.** One can easily see that  $D^2(m) = (GL(m))^1$ . Therefore, we can consider the algebraic coordinates on  $D^2(m)$  which give rise to the *algebraic coordinates on the total space*  $B^2(M)$ . Namely, if  $(U, u)$  is a coordinate chart on  $M$ , and  $(u^k, u_i^k, u_{ij}^k)$  are the corresponding natural coordinates on  $(\pi^2)^{-1}(U)$ , for the *algebraic coordinates* on  $(\pi^2)^{-1}(U)$  we take

$$(27) \quad (u^k, p_i^k, p_{ij}^k) \text{ where } p_i^k = u_i^k, p_{ij}^k = \tilde{u}_s^k u_{ij}^s.$$

In fact, we change coordinates on the second factor of  $U \times D^2(m) \cong (\pi^2)^{-1}(U)$ .

With respect to the algebraic coordinates the first prolongation  $P^1$  of an integrable  $G$ -structure  $P$  is described in the following way:

$$(28) \quad P^1|_U = \{(u^k, p_i^k, p_{ij}^k) \mid \|p_i^k\| \in G, \|p_{ij}^k\| \in \mathfrak{g}^1\}.$$

If we have two coordinate charts of the atlas  $\mathcal{A}$  on  $M$  and  $v^k = v^k(u^i)$  is the coordinate change, the corresponding gluing map (19) of trivializing charts of  $P^1$  is written with respect to the algebraic coordinates as follows:

$$(29) \quad g : U \cap U' \rightarrow G^1, \quad g(x) = \left( \frac{\partial v^k}{\partial u^i}(u(x)), \frac{\partial u^k}{\partial v^s}(v(x)) \frac{\partial v^s}{\partial u^i \partial u^j}(u(x)) \right).$$

### 3.4. $P^1(M)$ in terms of $P(M)$ .

## 4. FIRST PROLONGATION OF $G$ -STRUCTURE AND ASSOCIATED BUNDLES.

Let us now consider the constructions of the previous section for cases  $k = 1$  and  $k = 2$ .

The case  $k = 1$  is rather simple. We have  $D^1(m) \cong GL(m)$  and  $B^1(M)$  is the coframe bundle of  $M$ , because to each  $j_x^1 f$  we can put in correspondence the coframe  $\{f^* du^i\}$  at  $x \in M$ .

So we will consider in details the case  $k = 2$ .

**4.1. First prolongation of the Lie subgroup  $G \subset GL(m)$ .** Now let us consider the set  $\tilde{D}^2(m) = (\varphi_i^k, \varphi_{ij}^k)$ , where  $\|\varphi_i^k\|$  is an invertible matrix, and  $\varphi_{ij}^k$  are not necessarily symmetric with respect to lower indices. Then, the set  $\tilde{D}$  endowed with the operation  $*$  given by (16) is a Lie group called the *nonholonomic differential group of second order*. We will call it *first prolongation of the group  $GL(m)$*  and denote by  $GL^{(1)}(m)$ .

On the group  $GL^{(1)}(m)$  we can introduce another coordinate system:

$$(30) \quad g_j^k = \varphi_j^k, \quad a_{ij}^k = \tilde{\varphi}_s^k \varphi_{ij}^s$$

Then, with respect to these coordinates, by (16), we see that the product  $*$  can be written as follows:

$$(31) \quad (g_i^k, a_{ij}^k) * (h_i^k, b_{ij}^k) = (g_s^k h_i^s, \tilde{h}_s^k a_{pq}^s h_i^p h_j^q + b_{ij}^k).$$

These formulas can be written in a matrix form. To do this, we consider  $a_{ij}^k$  as a map  $a : \mathbb{R}^m \rightarrow \mathfrak{gl}(m)$ ,  $w^k \mapsto a_{ij}^k w^j$ . Then, (31) takes the form

$$(32) \quad (g, a) * (h, b) = (gh, adh^{-1}a \circ h + b).$$

Therefore, the group

$$GL^{(1)} \cong (GL(m) \times Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), *),$$

where  $*$  is defined in (32).

**Remark 4.** The vector space  $Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m))$  is a right  $GL(m)$ -module with respect to the action

$$(33) \quad \forall g \in GL(m), \quad a \in Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), \quad g \cdot a = adg^{-1}ag.$$

The group  $(GL(m) \times Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m)), *)$  is the extension of the group  $GL(m)$  with the right  $GL(m)$ -module  $Hom_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{gl}(m))$ . This fact motivates the definition of first prolongation for a subgroup  $G \subset GL(m)$ .



**Remark 5.** The same considerations can be done for the holonomic jet group  $D^2(m)$ .

4.1.1. *First prolongation of a Lie subgroup  $G \subset GL(m)$ .* The considerations of the previous subsection motivate

**Definition 3.** Let  $G \subset GL(m)$  be a Lie subgroup. Then the *first prolongation* of  $G$  is the group

$$(34) \quad G^{(1)} = G \times \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G))$$

with product

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2, \text{ad} g_2^{-1} a_1 g_2 + a_2).$$

**Remark 6.** In this case also the vector space  $\text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G))$  is a right  $G$ -module with respect to the action

$$(35) \quad \forall g \in G, \quad a \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)), \quad g \cdot a = \text{ad} g^{-1} a g.$$

The group  $(G \times \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)), *)$  is the extension of the group  $G$  with the right  $G$ -module  $\text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)), *)$ .

Therefore we have the short exact sequence of Lie groups:

$$0 \rightarrow \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathfrak{g}(G)) \rightarrow G^{(1)} \xrightarrow{\pi} G.$$

The Lie homomorphism  $\pi : G^{(1)} \rightarrow G$  comes from the natural projection of 2-jets onto 1-jets.

4.1.2. *Another coordinate system on  $B^2(M)$ .* Now recall that on  $D^2(m)$  we can take another coordinate system  $(g_i^k, a_{ij}^k)$  (see 30). With respect to the coordinates  $(g_i^k, a_{ij}^k)$ , the  $D^2(m)$ -action on  $B^2(M)$  is written as

$$(36) \quad h_i^k = \tilde{g}_s^k f_i^s, \quad h_{ij}^k = \tilde{g}_s^k f_{ij}^s - a_{lm}^k \tilde{g}_t^l \tilde{g}_r^m f_i^t f_j^r.$$

Let us find the expression for a trivialization

$$(\pi^2)^{-1}(U) \rightarrow U \times D^2(m)$$

with respect to the coordinates  $(g, a)$ . For this purpose we take a section  $s(x^i) = (x^i, \delta_i^k, 0)$  of  $B^2(M)$  over  $U$ , in fact this is the natural frame of order 2, this means that it consists of 2-jets of the coordinate functions. Then any point  $b^2 = (x^i, f_i^k, f_{ij}^k)$  can be written as  $b^2 = s(x) \cdot (g, a)$ , and the trivialization is given by

$$(37) \quad b^2 = (x^i, f_i^k, f_{ij}^k) \leftrightarrow (x, (g, a)).$$

Using (36), and the coordinate expression  $s(x)$ , we get

$$(38) \quad f_i^k = \tilde{g}_i^k, \quad f_{ij}^k = -a_{lm}^k \tilde{g}_i^l \tilde{g}_j^m.$$

Now let us write the gluing functions with respect to the coordinates  $(p_i^k, p_{ij}^k)$  on  $B^2(M)$  and  $(g_i^k, a_{ij}^k)$ . To do this we use the coordinate change (40) on  $B^2(M)$  and (30) on the group  $D^2(M)$ . We have

$$\begin{aligned} p_i^k &= \tilde{f}_i^k, & p_{ij}^k &= -f_{lm}^k \tilde{f}_i^l \tilde{f}_j^m \\ \bar{x}_i^k &= g_i^k, & \bar{x}_{ij}^k &= g_s^k a_{ij}^s \end{aligned}$$

and so,

$$(39) \quad f_i^k = \tilde{p}_i^k, \quad f_{ij}^k = -p_{lm}^k \tilde{p}_i^l \tilde{p}_j^m$$

$$(40) \quad (x^k, f_i^k, f_{ij}^k) \rightarrow (x^k, (p_i^k, p_{ij}^k)), \text{ where} \\ p_i^k = \tilde{f}_i^k, \quad p_{ij}^k = -f_{lm}^k \tilde{f}_i^l \tilde{f}_j^m.$$

Using these formulas, and (36), we can write the  $D^2(m)$ -action with respect to the coordinates  $(x^i, p_i^k, p_{ij}^k)$ . If  $(x^i, (p_i^k, p_{ij}^k)) \cdot (g, a) = (x^i, (q_i^k, q_{ij}^k))$ , then

$$(41) \quad q_i^k = p_s^k g_i^s, \quad q_{ij}^k = \tilde{g}_s^k p_{lm}^s g_i^l g_j^m + a_{ij}^k.$$

So, as it should be, at the second argument we get the product of elements of the group  $D^2(m)$  (cf. (31)). Hence, first of all from we get that

$$(42) \quad \tilde{p}_s^k g_i^s = \tilde{p}_i^k, \text{ and hence } \bar{p}_i^k = \tilde{g}_s^k p_i^s.$$

Now, from with (39), we get

$$-\bar{p}_{rt}^k \tilde{p}_l^r \tilde{p}_m^t g_i^l g_j^m + \tilde{p}_s^k g_t^s a_{ij}^t = f_{ij}^k,$$

hence follows

$$-\bar{p}_{lm}^k \tilde{p}_i^l \tilde{p}_j^m + \tilde{p}_s^k a_{ij}^s = f_{ij}^k = -p_{lm}^k \tilde{p}_i^l \tilde{p}_j^m.$$

Finally, we obtain

$$\bar{p}_{lm}^k = p_{lm}^k + \tilde{p}_t^k a_{lm}^t p_i^l p_j^m.$$

As the result, we get the following theorem.

**Theorem 1.**

Note that

$$(43) \quad x_i^k = \tilde{x}_i^k,$$

With this notation, we have

$$(44) \quad \bar{f}_i^k =$$

## 5. PROLONGATION OF $G$ -STRUCTURE

**5.1. First prolongation of integrable  $G$ -structure.** Let  $P(M, G)$  be an integrable  $G$ -structure, this means that there exists an atlas  $(U_\alpha, u_\alpha)$  such that  $\left\{ \frac{\partial}{\partial u_\alpha} \right\}$  are the sections of  $P$ .

A first prolongation of  $P(M, G)$  is the subbundle in  $B^2(M)$  with the total space

$$(45) \quad P^1(M) = \left\{ j_x^2 f \mid \left( \frac{\partial(f \circ u^{-1})^k}{\partial u^i} \Big|_{u(x)} \right) \in G \right\}$$

and the structure group  $G^1 \subset D^2(m)$  (the holonomic prolongation of  $G$ ):

$$(46) \quad G^1 = \left\{ j_0^2 \varphi \in D^2(m) \mid \left( \frac{\partial \varphi^k}{\partial u^i} \right) \in G \right\}.$$

The product in  $G^1$  is induced by the chain rule: if  $j_0^k \varphi = (\varphi_i^k, \varphi_{i_1 i_2}^k, \dots, \varphi_{i_1, \dots, i_k}^k)$   $j_0^k \psi = (\psi_i^k, \psi_{i_1 i_2}^k, \dots, \psi_{i_1, \dots, i_k}^k)$ , then

$$(47) \quad \eta_i^k = \psi_s^k \varphi_i^s, \quad \eta_{ij}^k = \psi_{pq}^k \varphi_i^p \varphi_j^q + \psi_s^k \varphi_{ij}^s, \dots$$

## 5.2. Prolongation of $G$ -structure.

5.2.1. *Nonholonomic jet bundle.* A  $k$ -th order nonholonomic jet is a “nonsymmetric” Taylor series and is defined by coordinates:

$$(48) \quad f_0^j = (f^j, f_i^j, f_{i_1 i_2}^j, \dots, f_{i_1, i_2, \dots, i_k}^j),$$

where  $f_{i_1 \dots i_j}^j$  are not supposed to be symmetric with respect to lower indices.

5.3. **Nonholonomic differential group.** The nonholonomic  $k$ -th order differential group is

$$D^{(k)}(m) = \{\varphi_0^k = (\varphi^j, \varphi_i^j, \varphi_{i_1 i_2}^j, \dots, \varphi_{i_1, i_2, \dots, i_k}^j) \mid \det(\varphi_i^j) \neq 0\},$$

and the product is also “induced by the chain rule”: if  $\eta_0^k = \psi_0^k \cdot \varphi_0^k$  and  $\varphi_0^k = (\varphi_i^j, \varphi_{i_1 i_2}^j, \dots, \varphi_{i_1, \dots, i_k}^j)$ ,  $\psi_0^k = (\psi_i^j, \psi_{i_1 i_2}^j, \dots, \psi_{i_1, \dots, i_k}^j)$ , then  $\eta_i^j = \psi_s^j \varphi_i^s$ ,  $\eta_{i_1 i_2}^j = \psi_{p_1 p_2}^j \varphi_{i_1}^{p_1} \varphi_{i_2}^{p_2} + \psi_s^j \varphi_{i_1 i_2}^s, \dots$

It is clear that  $D^k(m)$  is a Lie subgroup of  $D^{(k)}(m)$ , so we have the left  $D^k(m)$ -action on  $D^{(k)}(m)$ .

A *nonholonomic  $k$ -th order jet bundle* is the locally trivial bundle over  $M$  with the fiber  $D^{(k)}(m)$  associated with the  $D^k(m)$ -principal bundle  $B^k(M)$  with respect to the left  $D^k(m)$ -action on  $D^{(k)}(m)$ .

5.4. **First prolongation of arbitrary  $G$ -structure.** Let  $P(M, G)$  be a  $G$ -structure. A *first prolongation* of  $P(M, G)$  is the subbundle in  $B^{(2)}(M)$  with the total space

$$(49) \quad P^{(1)}(M) = \{f_0^2 \in B^{(2)}(M) \mid (f_i^j) \in G\}$$

and the structure group  $G^{(1)} \subset D^{(2)}(m)$  (the nonholonomic prolongation of  $G$ ):

$$(50) \quad G^{(1)} = \{\varphi_0^2 \in D^2(m) \mid (\varphi_i^k) \in G\}.$$

## 6. FIRST PROLONGATION OF $G$ -STRUCTURE IN TERMS OF COFRAME BUNDLE

The first prolongation  $P^{(1)}$  of  $P$  can be expressed in terms of  $P$  in the following ways:

$$(51) \quad P^{(1)} = \{p^1 : \mathbb{R}^m \rightarrow T_p P \mid \theta_p \alpha = 1_{\mathbb{R}^m}\},$$

or

$$(52) \quad P^{(1)} = \{\omega : T_p P \rightarrow \mathfrak{g} \mid \omega \sigma_p = 1_{\mathfrak{g}}\}$$

or

$$(53) \quad P^{(1)} = \{H_p \mid H_p \oplus V_p = T_p P\}.$$

We will mainly use the first representation (51), but note that the third representation (52) says that, geometrically,  $P^1$  consists of tangent subspaces transversal to vertical subspaces, i. e., of connections.

The projection  $\pi_0^1 : P^1 \rightarrow P$ , is defined in terms of (51) as follows  $(p^1 : \mathbb{R}^m \rightarrow T_p P) \mapsto b$ .

**Theorem 2** (Algebraic structure of  $G^{(1)}$ ).  *$G^{(1)}$  is isomorphic to the extension of  $G$  via the  $G$ -module  $\mathcal{L}(\mathbb{R}^n, \mathfrak{g})$ :*

$$(54) \quad G^{(1)} = G \times \mathcal{L}(\mathbb{R}^n, \mathfrak{g}), \quad (g_1, a_1) * (g_2, a_2) = (g_1 g_2, \text{ad} g_2^{-1} a_1 + a_2).$$

Action of  $G^{(1)}$  on  $P^{(1)}$  is described in the following way: for  $b^1 : \mathbb{R}^m \rightarrow T_b P$ , and  $g^1 = (g, a) \in G^1$ :

$$(55) \quad R_{g^1}^{(1)} b^1 = dR_g(b^1 \circ g) + \sigma_{pg} \circ a \circ g$$

## 7. FIRST PROLONGATION OF EQUIVARIANT MAP

**7.1. First prolongation of a  $G$ -space.** Let  $V$  be a manifold, then the first prolongation of  $V$  is

$$(56) \quad V^{(1)} = \{v^1 : \mathbb{R}^m \rightarrow T_v V \mid v \in V\},$$

Let  $\rho : G \times V \rightarrow V$  be a left action, then the first prolongation of  $\rho$  is

$$(57) \quad \rho^{(1)} : G^{(1)} \times V^{(1)} \rightarrow V^{(1)}, \quad \rho^{(1)}(g^1, v^1) = [dL_g \circ v^1 + \sigma_{gv} \circ A] \circ g^{-1}.$$

**7.2. Prolongation of equivariant map.**  $f : P \rightarrow V$  is an equivariant map.

Prolongation of  $f$  is  $f^{(1)} : P^{(1)} \rightarrow V^{(1)}$ ,  $f^{(1)}(b^1) = df_{\pi^1(b^1)} \circ b^1$ .

An equivariant map  $f : P \rightarrow V$  determines a section  $s : M \rightarrow E$ , where  $\pi_E : E \rightarrow M$  is a bundle with standard fiber  $V$  associated with  $P$ .

$f^1 : P^1 \rightarrow V^1$  maps  $b^1 \in P^1$  to the coordinates of  $(s(\pi(b)), (\nabla s)(\pi(b)))$  with respect to  $b$ , where  $\nabla$  corresponds to  $b^1$ .

**Example 1** (Simple example: vector field on  $\mathbb{R}^n$ ).  $V$  is a vector field on  $\mathbb{R}^m$ . The corresponding equivariant map is

$$(58) \quad f : B(\mathbb{R}^m) \rightarrow \mathbb{R}^m, b = \{\eta^a\}, f(b) = \{\eta^a(V(\pi(b)))\}.$$

Then the first prolongation of  $f$  is defined as follows. If  $b^{(1)} : e_i \in \mathbb{R}^m \rightarrow \frac{\partial}{\partial x^i} + \Gamma_{ij}^k \frac{\partial}{\partial x_j^k}$ , then

$$(59) \quad f^{(1)}(b^{(1)}) = (V, \nabla(\omega)V) = (V^i, \partial_j V^i + \Gamma_{js}^k V^s).$$

Action of  $G^1$  is written as follows

$$(60) \quad (g, a)(V, \nabla(\omega)V) = (\tilde{g}_i^k V^i, \tilde{g}_s^k \nabla_m V^s g_i^m + \tilde{g}_s^k a_{mj}^s V^j g_i^m),$$

At  $x_0$  such that  $V(x_0) = 0$ , we get the action

$$(61) \quad (g, a)(0, \nabla(\omega)V) = (0, \tilde{g}_s^k \nabla_m V^s g_i^m) = (0, \tilde{g}_s^k \partial_m V^s g_i^m)$$

Therefore, the prolonged action coincides with the action of the group  $GL(n)$  on the vector space of  $n \times n$ -matrices by conjugation. The invariants of this action are well known, for example, these are the trace and the determinant. Therefore, to find the invariants of a zero  $x_0$  of a vector field  $V$  we have to find the matrix  $[\partial_i V^j(x_0)]$  and then write the invariants of this matrix under the conjugation, for example, one of them is  $\det[\partial_i V^j(x_0)]$ .

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